Journal of Sound and Vibration (1998) **216**(1), 93–105
Article No. sv981689

NON-LINEAR VIBRATIONS OF VISCOELASTIC MOVING BELTS, PART II: FORCED VIBRATION ANALYSIS

L. ZHANG AND J. W. ZU

Department of Mechanical and Industrial Engineering, University of Toronto, 5 King's College Road, Toronto, Ontario, Canada M5S 3G8

(*Received* 7 *November* 1997, *and in final form* 31 *March* 1998)

The non-linear forced fibration of viscoelastic moving belts excited by the eccentricity of pulleys is investigated. The generalized equations of motion are derived for a viscoelastic moving belt with geometric non-linearities by adopting the linear viscoelastic differential constitutive law. The method of multiple scales is applied directly to the governing equations which are in the form of continuous non-autonomous gyroscopic systems. The amplitude of near- and exact-resonant steady state response for non-autonomous systems is predicted. The results obtained with quasi-static assumption and those without this assumption are compared. Effects of elastic and viscoelastic parameters, axial moving speed, and the geometric non-linearity on the system response are also studied.

© 1998 Academic Press

1. INTRODUCTION

Moving belts used in power transmission are an example of a class of mechanical systems commonly referred to as axially moving materials. The forced vibration analysis of such a system has been studied extensively. For linear vibration analysis of an axially moving string, the classical modal analysis applied to the linear non-translating string model is not directly applicable. Wickert and Mote [1] modified the classical modal analysis method by casting the equations of motion for a travelling string into a canonical, first order form and got the response in closed forms for axially moving materials subjected to arbitrary excitation and initial conditions.

For non-linear forced vibration of moving materials, Bapat and Srinivasan [2] used the method of harmonic balance to obtain the approximate form of the period–tension relationship. Naguleswaran and Williams [3] showed that a Mathieu–Hill type of system exists for a moving band or belt. Mote [4] predicted the stable–unstable boundaries by the application of Hsu's method. The frequency–amplitude relationship of the non-linear plane motion of travelling cables with the Lagrangian strain is obtained by Luongo *et al*. [5] using Galerkin method. Perkins and Mote [6] studied three-dimensional vibration of travelling elastic cables. Wang [7] analyzed parametric instabilities in belt and chain systems and the periodic tension fluctuations which are caused by the impulsive forces. The dynamics stability of a moving string under three-dimensional vibration is investigated by Huang *et al*. [8]. More recently, Moon and Wickert [9] developed a modal perturbation solution in the context of the asymptotic method fo Krylov, Bogoliubov, and Mitropolsky for a continuous, non-autonomous and gyroscopic system with geometric non-linearity. Near- and exact-resonant response amplitudes were predicted by the approach.

© 1998 Academic Press

In all of the work mentioned above, the belt is assumed to be linear elastic and damping is either ignored or introduced simply as linear viscous without reference to any damping mechanism. However, belts are usually composed of some metallic or ceramic reinforcement materials like steel-cord or glass-cord and polymeric materials such as rubber. Most of these materials exert inherently viscoelastic behavior, i.e., they flow when subjected to stress or strain. Such flow is accompanied by the dissipation of energy due to some internal loss. To accurately describe the material property of moving belts, viscoelastic constitutive relation should be employed.

Various methods have been developed for the vibration analysis of viscoelastic structures. Findley *et al*. [10] used the correspondence and superposition principles to solve the governing equations of the viscoelastic beams. Chen [11] used Laplace transform and the resulting equation was solved by the finite element method. Fung *et al*. [12] employed Galerkin approximation and reduced the governing equation to a third order non-linear ordinary differential equation. The Stevens method was followed to analyze the stability of the linear system. The method of variation of parameters and the method of averaging were used to analyze the dynamic response of non-linear systems. The Routh–Hurwitz criterion was adopted to investigate the stability of steady solutions of the parametric resonance and the non-linear effects.

There is only one paper by Fung *et al*. [13] so far discussing the dynamic response of a viscoelastic moving string. In the paper, the string material was assumed to be constituted by the hereditary integral type. The governing equation was reduced to a set of second order non-linear differential–integral equations and the resulting equations were solved by the finite difference method.

In this paper, the non-linear forced vibration of viscoelastic moving belts is studied. The linear viscoelastic differential constitutive law is employed to model the viscoelastic characteristic of belt materials. The governing equations of motion are derived for a viscoelastic moving belt with geometric non-linearities. The method of multiple scales is applied directly to the equations which are in the form of continuous non-autonomous gyroscopic systems. This direct treatment does not involve a prior assumption regarding the spatial solutions. The amplitude of near- and exact-resonant steady state response for non-autonomous systems is predicted. The results obtained with the quasi-static assumption and those without this assumption are compared. The effects of elastic and viscoelastic parameters, axial moving speed, and the geometric non-linearity on the system response are also investigated.

2. EQUATIONS OF MOTION

Figure 1 shows a prototypical system of a moving belt. c is the transport speed of the belt, r_0, r_1, e_0 and e_1 denote radii and eccentricities of the pulleys. The equation of motion

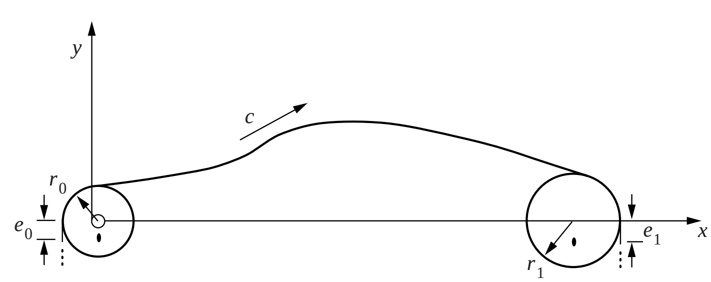

Figure 1. A prototypical model of a viscoelastic moving belt driven by eccentrically-mounted drive and pulleys.

in the y direction can be obtained by Newton's second law [13]:

$$\left(\frac{T}{A} + \sigma\right)V_{xx} + V_x\sigma_x = \rho\left(\frac{\partial^2 V}{\partial t^2} + 2c\,\frac{\partial^2 V}{\partial x\,\partial t} + c^2\,\frac{\partial^2 V}{\partial x^2}\right), \tag{1}$$

where the subscript notation x and t denote partial differentiation with respect to the spatial Cartesian co-ordinate x and time t, V is the displacement in the transverse direction, σ is the perturbed stress, A is the area of cross-section of the string, ρ is the mass per unit volume, and T is the initial force.

The system is subjected to the non-homogeneous boundary conditions

$$V(0, t) = e_0 \sin(\Omega_0 t), \qquad V(L, t) = e_1 \sin(\Omega_1 t), \tag{2}$$

where L is the length of the belt span, and Ω_0, Ω_1 are rotational frequencies of the pulleys. The substitution of displacement V with $V + e_0 \sin(\Omega_0 t) + (e_1 \sin(\Omega_1 t) - e_0 \sin(\Omega_0 t))(x/L)$ renders the boundary conditions homogeneous, and the excitation is transferred from the boundary to the domain [3]. Correspondingly, equations (1) and (2) are changed into the form

$$\left(\frac{T}{A} + \sigma\right)V_{xx} + V_x\sigma_x + \frac{F(x, t)}{A} = \rho\left(\frac{\partial^2 V}{\partial t^2} + 2c\,\frac{\partial^2 V}{\partial x\,\partial t} + c^2\,\frac{\partial^2 V}{\partial x^2}\right), \tag{3}$$

$$V(0, t) = 0, \qquad V(L, t) = 0. \tag{4}$$

Note that $F(x, t)$ is the external force per unit length which is transferred from the boundary support excitation.

In this paper, only the eccentricity of the right pulley is considered, and thus the external force $F(x, t)$ can be expressed as

$$F(x, t) = (x - 2ir_1)e_1\Omega^2\,\mathrm{e}^{i\Omega t} + cc, \tag{5}$$

where Ω is the rotational frequency of the right hand pulley, and cc denotes the complex conjugate of all preceding terms on the right side of equation (5). Under the assumption of no slip, the relation between the excitation frequency and the transport speed is of the form

$$\Omega = c/r_1. \tag{6}$$

The one-dimensional linear differential viscoelastic constitutive law can be written as

$$\sigma(t) = E^*\varepsilon(t), \tag{7}$$

where the linear differential operator E^* is determined by the viscoelastic property of belt materials and may be handled formally as an algebraic quantity.

Applying the linear differential viscoelastic constitutive law, equation (7), and considering the Lagrangian strain component, the perturbed stress is in the form

$$\sigma = E^*(\tfrac{1}{2}V_x^2). \tag{8}$$

Substituting equation (8) into equation (3) yields

$$\rho\,\frac{\partial^2 V}{\partial t^2} + 2\rho c\,\frac{\partial^2 V}{\partial t\,\partial x} + \left(\rho c^2 - \frac{T}{A}\right)\frac{\partial^2 V}{\partial x^2} = E^*(\tfrac{1}{2}V_x^2)V_{xx} + V_x\{E^*(\tfrac{1}{2}V_x^2)\}_x + \frac{F(x, t)}{A}. \tag{9}$$

Introducing the following non-dimensional parameters,

$$v = \frac{V}{L}, \qquad \xi = \frac{x}{L}, \qquad \tau = t\left(\frac{T}{\rho A L^2}\right)^{1/2}, \qquad \gamma = c\left(\frac{\rho A}{T}\right)^{1/2},$$

$$E = \frac{E^* A}{T}, \qquad f(x, t) = \frac{F(x, t)L}{T},$$

the following non-dimensional equation of transverse motion can be obtained:

$$\frac{\partial^2 v}{\partial \tau^2} + 2\gamma \frac{\partial^2 v}{\partial \tau\, \partial \xi} + (\gamma^2 - 1)\frac{\partial^2 v}{\partial \xi^2} = N(v) + f(x, t), \tag{10}$$

where the non-linear operator $N(v)$ is defined as

$$N(v) = E(\tfrac{1}{2}v_\xi^2)v_{\xi\xi} + v_\xi\{E(\tfrac{1}{2}v_\xi^2)\}_\xi. \tag{11}$$

Equations (12) and (13) are the generalized equations of motion valid for all kinds of viscoelastic models. As a first step, the most frequently used Kelvin viscoelastic model is chosen to describe the viscoelastic property of the belt material in this paper. The corresponding linear dimensionless differential operator E for Kelvin viscoelastic model is

$$E = E_e + E_v \frac{\partial}{\partial \tau}, \tag{12}$$

where

$$E_e = \frac{E_0 A}{T}, \qquad E_v = \eta\sqrt{\frac{A}{\rho T L^2}}. \tag{13, 14}$$

E_0 is the stiffness constant of the spring and η is the dynamic viscosity of the dashpot.

Substituting equation (12) into equation (11), and with some manipulations, the non-liear operator $N(v)$ for the Kelvin viscoelastic model becomes

$$N(v) = \tfrac{3}{2}E_e v_\xi^2 v_{\xi\xi} + E_v \frac{\partial}{\partial \tau}(\tfrac{1}{2}v_\xi^2)v_{\xi\xi} + v_\xi E_v \frac{\partial}{\partial \tau}(v_\xi v_{\xi\xi}), \tag{15}$$

where the first term on the right side of equation (15) is a non-linear term related to elasticity and the last two terms are non-linear terms related to viscoelasticity.

Introduce the mass, gyroscopic, and linear stiffness operators as follows:

$$M = I, \qquad G = 2\gamma\frac{\partial}{\partial \xi}, \qquad K = (\gamma^2 - 1)\frac{\partial^2}{\partial \xi^2}, \tag{16}$$

where operators M and K are symmetric and positive definite for sub-critical transport speeds and G is skew-symmetric and represents a convective Coriolis acceleration component. Thus, equation (10) can be written in a standard symbolic form

$$M v_{\tau\tau} + G v_\tau + K v = N(v) + f(x, t). \tag{17}$$

3. NON-LINEAR FORCED VIBRATION ANALYSIS

In this section, the amplitude of near- and exact-resonant steady state response for non-autonomous systems is predicted. The method of multiple scales [14] is applied directly to the equations which are in the form of continuous non-autonomous gyroscopic systems. Introducing a small dimensionless parameter ε as a bookkeeping device, equation (17) can be rewritten as

$$Mv_{\tau\tau} + Gv_\tau + Kv = \varepsilon N(v) + \varepsilon f(\xi, \tau). \quad (18)$$

A second order uniform approximation is sought in the form

$$v(\xi, \tau, \varepsilon) = v_0(\xi, T_0, T_1) + \varepsilon v_1(\xi, T_0, T_1) + \cdots . \quad (19)$$

Substituting equation (19) and the time derivatives in terms of T_0 and T_1 into equation (18) and equating coefficients of like powers of ε,

$$M\frac{\partial^2 v_0}{\partial T_0^2} + G\frac{\partial v_0}{\partial T_0} + Kv_0 = 0, \quad (20)$$

$$M\frac{\partial^2 v_1}{\partial T_0^2} + G\frac{\partial v_1}{\partial T_0} + Kv_1 = -2M\frac{\partial^2 v_0}{\partial T_0\,\partial T_1} - G\frac{\partial v_0}{\partial T_1} + N(v_0) + f(\xi, \tau). \quad (21)$$

All the excitation components on the right side of equation (21) except for $f(\xi, \tau)$ are evaluated from first order solution v_0.

The solution of equation (20) is of the form

$$v_0 = \psi_k(\xi)A_k(T_1)\,\mathrm{e}^{i\omega_k T_0} + \bar{\psi}_k(\xi)\bar{A}_k(T_1)\,\mathrm{e}^{-i\omega_k T_0}, \quad (22)$$

where the overbar denotes complex conjugate, ω_k is the kth natural frequency, and ψ_k is the kth eigenfunction. For linear moving belts, ω_k and ψ_k are given by [1]

$$\omega_k = k\pi(1 - \gamma^2), \qquad \psi_k = \sqrt{2}\sin(k\pi\xi)\,\mathrm{e}^{(ik\pi\gamma\xi)}. \quad (23, 24)$$

Function A_k is in equation (22) will be determined by eliminating the secular terms from v_1. Substituting equation (22) into equation (21) leads to

$$M\frac{\partial^2 v_1}{\partial T_0^2} + G\frac{\partial v_1}{\partial T_0} + Kv_1 = M_{1k}(E_e + 2i\omega_k E_v)A_k^3\,\mathrm{e}^{3i\omega_k T_0}$$

$$+[M_{2k}(3E_e + 2i\omega_k E_v)A_k^2\bar{A}_k - 2i\omega_k A_k' M\psi_k - A_k' G\psi_k]\,\mathrm{e}^{i\omega_k T_0} + f_0(\xi)\,\mathrm{e}^{i\Omega T_0} + cc, \quad (25)$$

where the prime indicates the derivative with respect to T_1, and M_{1k}, M_{2k} are non-linear spatial operators which are defined as

$$M_{1k} = \frac{3}{2}\left(\frac{\partial\psi_k}{\partial\xi}\right)^2\frac{\partial^2\psi_k}{\partial\xi^2}, \qquad M_{2k} = \frac{1}{2}\left[\left(\frac{\partial\psi_k}{\partial\xi}\right)^2\frac{\partial^2\bar{\psi}_k}{\partial\xi^2} + 2\frac{\partial\psi_k}{\partial\xi}\frac{\partial\bar{\psi}_k}{\partial\xi}\frac{\partial^2\psi_k}{\partial\xi^2}\right]. \quad (26, 27)$$

The solvability condition requires that the right side of equation (25) be orthogonal to every solution of the homogeneous problem. For the case where internal resonance does not exist, the solvability condition can be determined as

$$-2i\omega_k A_k' m_k - A_k' g_k i + (3E_e + 2i\omega_k E_v)A_k^2\bar{A}_k m_{2k} + f_k\,\mathrm{e}^{i(\Omega - \omega_k)T_0} = 0 \quad (28)$$

in which

$$m_k = \langle M\psi_k, \psi_k\rangle, \qquad g_k = -i\langle G\psi_k, \psi_k\rangle, \qquad m_{2k} = \langle M_{2k}, \psi_k\rangle, \qquad f_k = \langle f_0, \psi_k\rangle \quad (29\text{–}32)$$

and the notation $\langle \cdot , \cdot \rangle$ represents the standard inner product of two complex functions over $\xi \in (0, 1)$. Substituting the natural frequencies and eigenfunctions in equations (23) and (24) into equations (29)–(32) gives

$$m_k = 1, \qquad g_k = 2k\pi\gamma^2, \qquad m_{2k} = -\tfrac{1}{4}\pi^4 k^4(3 + 2\gamma^2 + 3\gamma^4). \tag{33–35}$$

Note that m_{2k} and g_k are real. In the near and exact-resonance cases, introduce a detuning parameter $\mu = o(1)$ defined by

$$\Omega = \omega_k + \epsilon\mu. \tag{36}$$

Substituting equation (36) into equation (28) leads to

$$-2i\omega_k A_k' m_k - A_k' g_k i + (3E_e + 2i\omega_k E_v)A_k^2 \overline{A}_k m_{2k} + f_k\, e^{i\mu T_1} = 0. \tag{37}$$

For convenience, express A_k in the polar form

$$A_k = \tfrac{1}{2}\alpha_k\, e^{i\beta_k}. \tag{38}$$

Note that α_k and β_k represent the amplitude and the phase angle of the response, respectively. Substituting equation (38) into equation (37) and separating the resulting equation into real and imaginary parts yields

$$\tfrac{1}{2}\alpha_k\beta_k'(2\omega_k m_k + g_k) + \frac{3\alpha_k^3 E_e m_{2k}}{8} = -\mathrm{Re}\,(f_k)\cos(\mu T_1 - \beta_k) + \mathrm{Im}\,(f_k)\sin(\mu T_1 - \beta_k), \tag{39}$$

$$-\tfrac{1}{2}\alpha_k'(2\omega_k m_k + g_k) + \frac{\alpha_k^3\omega_k E_v m_{2k}}{4} = -\mathrm{Re}\,(f_k)\sin(\mu T_1 - \beta_k) - \mathrm{Im}\,(f_k)\cos(\mu T_1 - \beta_k). \tag{40}$$

Since T_1 appears explicitly in equations (39) and (40), the equations are called a non-autonomous system. It is convenient to eliminate the explicit dependence on T_1, thereby transforming these equations into an autonomous system. This can be accomplished by introducing a new dependent variable θ_k defined by

$$\theta_k = \mu T_1 - \beta_k. \tag{41}$$

Using equation (41), equations (39) and (40) can be rewritten as

$$\tfrac{1}{2}\alpha_k(\mu - \theta_k')(2\omega_k m_k + g_k) + \frac{3\alpha_k^3 E_e m_{2k}}{8} = -\mathrm{Re}\,(f_k)\cos(\theta_k) + \mathrm{Im}\,(f_k)\sin(\theta_k), \tag{42}$$

$$-\tfrac{1}{2}\alpha_k'(2\omega_k m_k + g_k) + \frac{\alpha_k^3\omega_k E_v m_{2k}}{4} = -\mathrm{Re}\,(f_k)\sin(\theta_k) - \mathrm{Im}\,(f_k)\cos(\theta_k). \tag{43}$$

For the steady state response, the amplitude α_k and the new phase angle θ_k in equations (42) and (43) should be constant. Thus, setting $\alpha_k' = 0$, $\theta_i' = 0$ and with some manipulations, the amplitude and phase of the response can be determined from the algebraic equations

$$c_3(\alpha_k^2)^3 + c_2(\alpha_k^2)^2 + c_1(\alpha_k^2) + c_0 = 0, \tag{44}$$

$$\theta_k = \tan^{-1}\left(\frac{C_k\,\mathrm{Re}\,(f_k) - (1 + D_k)\,\mathrm{Im}\,(f_k)}{(1 + D_k)\,\mathrm{Re}\,(f_k) + C_k\,\mathrm{Im}\,(f_k)}\right), \tag{45}$$

where

$$C_k = \frac{\omega_k E_v m_{2k}}{2(2\omega_k m_k + g_k)}, \qquad D_k = \frac{3E_e m_{2k}}{4(2\omega_k m_k + g_k)}, \tag{46, 47}$$

$$c_0 = -((\mathrm{Re}\,(f_k))^2 + (\mathrm{Im}\,(f_k))^2), \qquad c_1 = \tfrac{1}{4}(2\omega_k m_k + g_k)^2\mu^2, \tag{48, 49}$$

$$c_2 = \frac{3E_e m_{2k}(2\omega_k m_k + g_k)\mu}{8}, \qquad c_3 = \tfrac{1}{64}((3E_e m_{2k})^2 + (2\omega_k E_v m_{2k})^2). \tag{50, 51}$$

Note that c_0, c_1 and c_2 are independent of the viscoelastic property of the belt material. Only c_3 changes with E_v which is a measure of the degree of viscoelastic behavior of the belt. Equation (44) has one real and two complex conjugate roots for moving speeds below a critical fold velocity, and three real roots above that point.

Substituting equations (23), (34) and (35) into equations (46)–(51) yields

$$C_k = -\tfrac{1}{16}\pi^4 k^4(1-\gamma^2)(3+2\gamma^2+3\gamma^4)E_v, \qquad D_k = -\tfrac{3}{32}E_e\pi^3 k^3(3+2\gamma^2+3\gamma^4), \tag{52, 53}$$

$$c_1 = k^2\pi^2\mu^2, \qquad c_2 = -\frac{3E_e k^5\pi^5}{16}(3+2\gamma^2+3\gamma^4)\mu, \tag{54, 55}$$

$$c_3 = \frac{9E_e^2 k^8\pi^8}{1024}(3+2\gamma^2+3\gamma^4)^2 + \frac{E_v^2 k^{10}\pi^{10}(1-\gamma^2)^2(3+2\gamma^2+3\gamma^4)^2}{256}. \tag{56}$$

If the "quasi-static stretch" is assumed, the steady response amplitude can also be obtained using the same equation (44) by simply redefining M_{2k} as

$$M_{2k} = \frac{\partial^2\psi_k}{3\partial\xi^2}\int_0^1\left(\frac{\partial\psi_k}{\partial\xi}\right)\left(\frac{\partial\overline{\psi}_k}{\partial\xi}\right)\mathrm{d}\xi + \frac{\partial^2\overline{\psi}_k}{6\partial\xi^2}\int_0^1\left(\frac{\partial\psi_k}{\partial\xi}\right)^2\mathrm{d}\xi. \tag{57}$$

Using the same definitions, m_{2k}, C_k and D_k can be derived as

$$m_{2k} = -\frac{k^2\pi^2}{6\gamma^2}(2k^2\pi^2\gamma^2(\gamma^2+1)^2 + \sin^2(k\pi\gamma)), \tag{58}$$

$$C_k = -\frac{E_v\pi^2 k^2(1-\gamma^2)(2\pi^2 k^2\gamma^2(1+\gamma^2)^2 + \sin^2(k\pi\gamma))}{24\gamma^2}, \tag{59}$$

$$D_k = -\frac{E_e\pi k(2\pi^2 k^2\gamma^2(1+\gamma^2)^2 + \sin^2(k\pi\gamma))}{16\gamma^2}. \tag{60}$$

Similar to the previous calculations, the corresponding coefficients of equation (44), c_1, c_2 and c_3, are given by

$$c_1 = k^2\pi^2\mu^2, \qquad c_2 = -\frac{E_e k^3\pi^3(2k^2\pi^2\gamma^2(1+\gamma^2)^2 + \sin^2(k\pi\gamma))\mu}{8}, \tag{61, 62}$$

$$c_3 = \frac{(9E_e^2 + 4\omega_k^2 E_v^2)k^4\pi^4(2k^2\pi^2\gamma^2(1+\gamma^2)^2 + \sin^2(k\pi\gamma))^2}{2304\gamma^4}. \tag{63}$$

The response of the first order approximation is obtained by substituting the root of equations (44) and (45) into equation (22) as

$$v_0 = \psi_k(\xi)\tfrac{1}{2}\alpha_k\,\mathrm{e}^{i(\Omega t - \theta_k)} + cc. \tag{64}$$

Using equation (37), the following relation can be obtained:

$$A_k' = \frac{(3E_e + 2i\omega_k E_v) m_{2k} A_k^2 \bar{A}_k + f_k \, e^{i\mu T_1}}{2i\omega_k m_k + ig_k}. \tag{65}$$

Substituting equation (65) into (25) yields

$$M \frac{\partial^2 v_1}{\partial T_0^2} + G \frac{\partial v_1}{\partial T_0} + K v_1 = f_1(\xi) A_k^3 \, e^{3i\omega_k T_0} + f_2(\xi, T_1) \, e^{i\omega_k T_0} + cc, \tag{66}$$

where

$$f_1(\xi) = M_{1k}(E_e + 2i\omega_k E_v), \tag{67}$$

$$f_2(\xi, T_1) = f(\xi) \, e^{i\mu T_1} + M_{2k}(3E_e + 2i\omega_k E_v) A_k^2 \bar{A}_k - (2i\omega_k M\psi_k + G\psi_k) \frac{(3E_e + 2i\omega_k E_v) m_{2k} A_k^2 \bar{A}_k + f_k \, e^{i\mu T_1}}{2i\omega_k m_k + ig_k}, \tag{68}$$

$$M_{1k} = \frac{3}{2} \left(\frac{\partial \psi_k}{\partial \xi} \right)^2 \frac{\partial^2 \psi_k}{\partial \xi^2}. \tag{69}$$

The solution of equation (66), which is the corresponding response correction of v_0, can be obtained using separation of variables,

$$v_1(T_0, T_1) = h_1(\xi) A_k^3 \, e^{3i\omega_k T_0} + h_2(\xi, T_1) A_k^2 \bar{A}_k \, e^{i\omega_k T_0} + cc, \tag{70}$$

where

$$h_1(\xi) = \sum_{n = \pm 1, \pm 2 \cdots} \frac{\langle f_1(\xi), \psi_n(\xi) \rangle}{1 - 3\omega_k/\omega_n} \psi_n(\xi),$$

$$h_2(\xi, T_1) = \sum_{\substack{n = \pm 1, \pm 2, \ldots \\ n \neq k}} \frac{\langle f_2(\xi, T_1), \psi_n(\xi) \rangle}{1 - \omega_k/\omega_n} \psi_n(\xi). \tag{71, 72}$$

From equations (70)–(72) it is evident that the spatial variations of the first order solutions are different from those of the linear solutions. Hence, the validity of the assumption that the spatial variation can be represented in terms of linear eigenfunctions is questionable. However, this assumption is adopted in the commonly used perturbation approach in which the partial differential equation is discretized first using linear eigenfunctions.

4. NUMERICAL RESULTS

In this section, numerical results of steady response amplitudes near and at exact resonance for moving belts are presented. Effects of the transport speed, nonlinearity and the viscoelastic parameter on the steady state response are discussed.

To compare the results obtained in this study with those given in reference [9], linear elastic constitutive law is first employed. Figure 2 shows the response amplitudes predicted by the method of multiple scales under the quasi-static assumption and those in reference [9]. The non-dimensional transport speed γ ranges from 0·1 to 0·4 which includes the resonant region. Three different values of the non-linear parameter E_e are chosen to

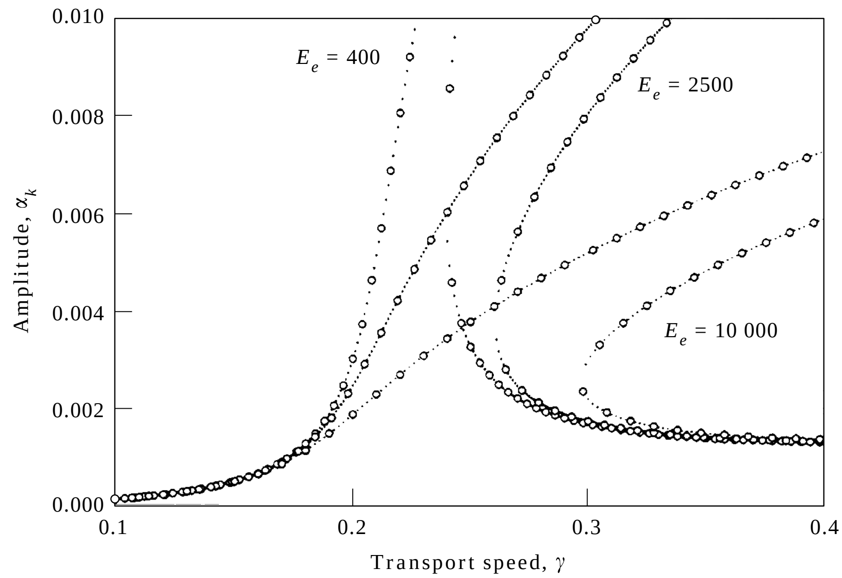

Figure 2. Comparison of response amplitudes predicted by the method of multiple scales and those given in reference [9] under the quasi-static assumption: ○, method of multiple scales; ——, given in reference [9].

investigate the non-linear effect. The system parameters are $e_1 = 0{\cdot}00083$, $r_1 = 0{\cdot}0733$. It is clear that the results obtained in this study are identical to those given in reference [9]. This shows the validity of the proposed method. It can be seen that the effect of the moving speed on the response amplitude is significant. This is because both the linear natural frequencies of the system and the excitation frequencies depend on the moving speed. For moving speeds below a critical speed, the amplitude is single valued; for moving speed above that critical speed, the response amplitude has three values corresponding to the same transport speed γ. Thus, the system shows a typical multi-valued non-linear phenomenon. When the excitation frequencies determined by the moving speed are near or at exact natural frequencies, the response amplitude becomes very large. In addition, it is observed that the bending of the curves is responsible for the jump phenomenon. The maximum amplitude is attainable only when approached from a lower moving speed. In the multi-valued response, the intermediate response is unstable and hence, cannot be produced both numerically and experimentally. However, the other two amplitudes are stable. Note that E_e is a measure of non-linearity. The higher the value of E_e, the stronger the non-linearity of the system. It can be seen that E_e has a significant effect on the steady response amplitude of the system. With the increase of E_e, response under the same transport speed decreases.

The response amplitudes obtained using the method of multiple scales without the quasi-static assumption are shown in Figure 3. The same system parameters as those in Figure 2 are adopted. It is clear that the results without the quasi-static assumption and those with such an assumption are close to each other over the non-resonance region. The difference, however, grows within the resonant region. This shows that the quasi-static assumption is accurate at most time span. However, since the near- and exact-resonant response is much larger, the differences between the results with the quasi-static assumption and those without quasi-static becomes significant.

The effects of the viscoelastic parameter E_v on the response amplitude are illustrated in Figures 4–6. The non-dimensional radius r_1 and eccentricity of pulley e_1 are 0·00083 and

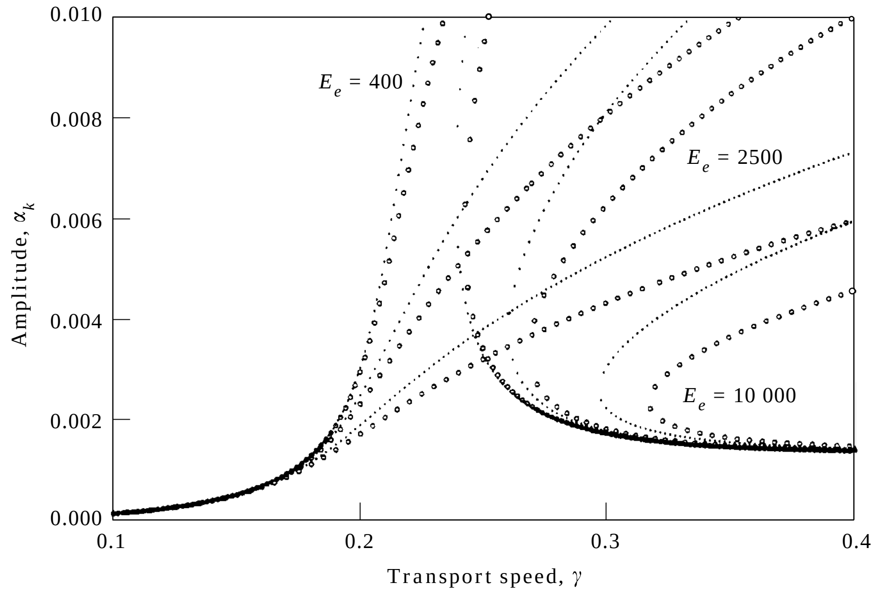

Figure 3. Comparison of response amplitudes without the quasi-static assumption and those with the quasi-static assumption: ○, without the quasi-static assumption; ——, with the quasi-static assumption.

0·0733. In Figure 4, $E_e = 1$. Three different values of E_v are chosen as 0·1, 25 and 50, respectively. From Figure 4, it is evident that the damping introduced by the viscoelastic model reduces the amplitude of response, especially at the near- and exact-resonant region. The amplitude of the response decreases as the damping increases. The maximum amplitude reduction for $E_v = 25$ is 40·3% while for $E_v = 50$ the maximum amplitude reduction is 55·6%. The degree of vibration reduction also depends on the non-linear parameter E_e. Figures 5 and 6 show the response amplitudes corresponding to higher

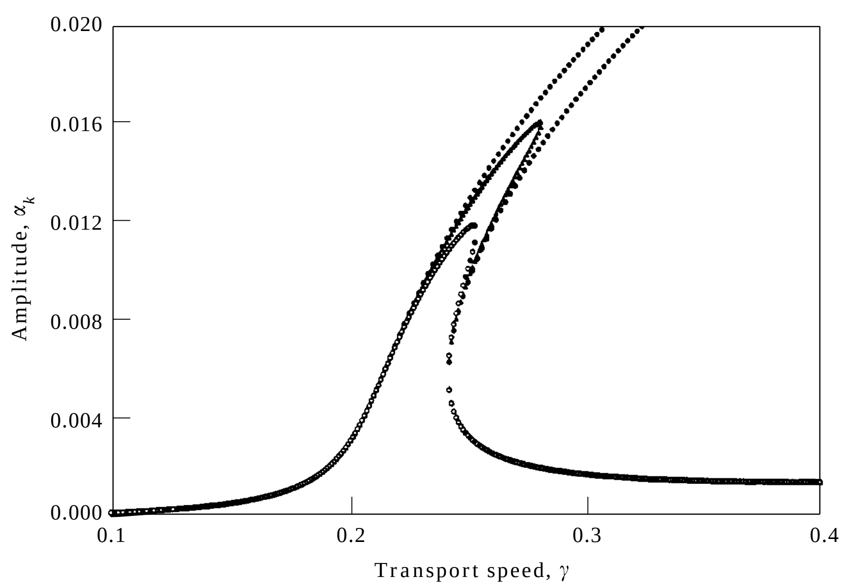

Figure 4. Comparison of response amplitudes of different E_v for $E_e = 400$: ●, $E_v = 0$; ○, $E_v = 25$; ▲, $E_v = 50$.

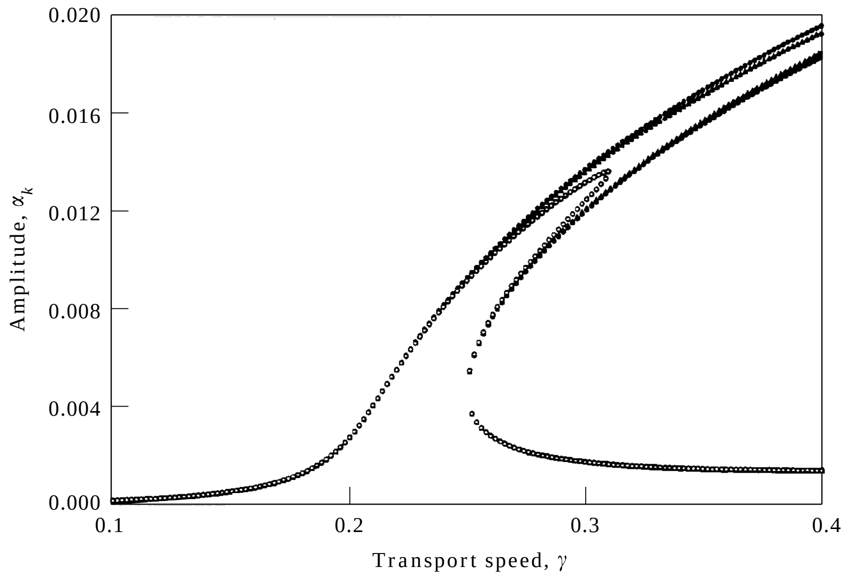

Figure 5. Comparison of response amplitudes of different E_v for $E_e = 800$: ●, $E_v = 0$; ○, $E_v = 25$; ▲, $E_v = 50$.

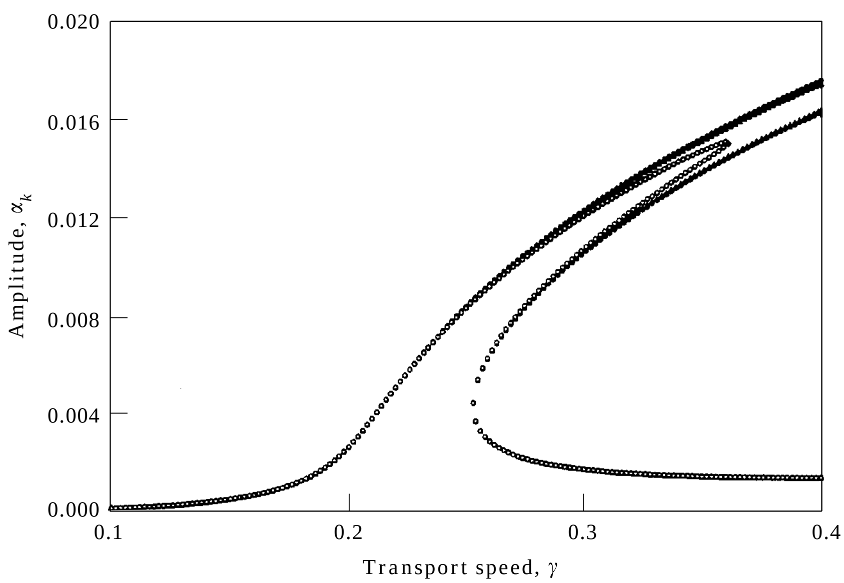

Figure 6. Comparison of response amplitudes of different E_v for $E_e = 1000$: ●, $E_v = 0$; ○, $E_v = 25$; ▲, $E_v = 50$.

values of E_e, i.e., $E_e = 800$ and $E_e = 1000$, respectively. It is seen that under the same E_v, the amplitude increases as E_e increases. Therefore, the degree of vibration reduction depends on the ratio E_v/E_e. When the ratio E_v/E_e is very small, the influence of viscoelasticity on vibration reduction is not significant.

5. CONCLUSIONS

The amplitude of near- and exact-resonant response is predicted for forced vibrations of viscoelastic moving belts excited by the eccentricity of pulleys. Based on the linear viscoelastic moving belts excited by the eccentricity of pulleys. Based on the linear viscoelastic differential constitutive law, the generalized equations of motion are derived for a viscoelastic moving belt with geometric nonlinearities. The method of multiple scales is applied directly to the governing equations which are in the form of continuous non-autonomous gyroscopic systems.

From the above study, the following conclusion can be drawn:

(1) The moving speed of belts has a significant effect on the steady state response since both the linear natural frequencies and the excitation frequencies depend on the moving speed. For moving speeds below a critical speed, the response amplitude is single valued; for moving speed above that critical speed, the response amplitude has three values corresponding to the same transport speed.

(2) The viscoelastic model can be used to accurately describe the damping mechanism of belt materials. The damping introduced by the viscoelastic model determines the vibration reduction. Therefore, it is possible to predict a desirable damping value that can significantly reduce the transverse vibration of moving belts.

(3) The method of multiple scales is applied directly to the governing equations. No assumptions regarding the spatial dependence of the motion are made while commonly used perturbation approaches assume that the motion of the non-linear system has the same spatial dependence as the linear system. Discrepancy between the approach proposed in this paper and those commonly used perturbation approaches appears at the first level approximation. The proposed approach can be generalized to other frequently used viscoelastic models such as the three parameter model and other gyroscopic systems with geometric non-linearity.

It should be mentioned that the viscoelastic property not only reduces the vibration, but also shifts stability boundaries significantly in the parametric excitation analysis of moving belts which will be shown in a future paper. Furthermore, a viscoelastic model can also be used to predict belt creep which leads to the excessive slip of the belt drive system. More work needs to be done to obtain understanding of the effects of the viscoelastic property of belts.

ACKNOWLEDGMENT

This research was financially supported by a research grant from the National Science and Engineering Research Council of Canada.

REFERENCES

1. J. A. Wickert and C. D. Mote Jr. 1990 *ASME Journal of Applied Mechanics* **57**, 738–744. Classical vibration analysis of axially moving continua.
2. V. A. Bapat and P. Srinivasan 1967 *ASME Journal of Applied Mechanics* **34**, 775–777. Non-linear transverse oscillation in traveling strings by the method of harmonic balance.
3. S. Naguleswaran and C. J. H. Williams 1968 *International Journal of Mechanical Sciences* **10**, 239–250. Lateral vibration of band-saw blades, pulley belts and the like.
4. C. D. Mote Jr 1968 *ASME Journal of Applied Mechanics* **35**, 171–172. Parametric excitation of an axially moving string.
5. A. Luongo, G. Rega and F. Vestroni 1984 *International Journal of Nonlinear Mechanics* **19**, 39–45. Planar nonlinear free vibrations of an elastic cable.
6. N. C. Perkins and C. D. Mote Jr 1987 *Journal of Sound and Vibration* **144**, 325–340. Three-dimensional vibration of travelling elastic cables.
7. K. W. Wang 1992 *ASME Journal of Vibration and Acoustics* **114**, 119–126. On the stability of chain drive systems under periodic sprocket oscillation.

8. J. S. Huang, R. F. Fung and C. H. Lin 1995 *International Journal of Mechanical Sciences* **37**, 145–160. Dynamic stability of a moving string undergoing three-dimensional vibration.
9. J. Moon and J. A. Wickert 1997 *Journal of Sound and Vibration* **200**, 419–431. Non-linear vibration of power transmission belts.
10. W. N. Findley, L. S. Lai and K. Onarna 1976 *Creep and Relaxation of Nonlinear Viscoelastic Materials*. New York: North-Holland. See p. 109.
11. T. M. Chen 1995 *International Journal for Numerical Method in Engineering* **38**, 509–522. The hybrid Laplace transform/finite element method applied to the quasi-static and dynamic analysis of viscoelastic Timoshenko beams.
12. R. F. Fung, J. S. Huang and W. H. Chen 1996 *Journal of Sound and Vibration* **198**, 1–6. Dynamic stability of a viscoelastic beam subjected to harmonic and parametric excitations simultaneously.
13. R. F. Fung, J. S. Huang and Y. C. Chen 1997 *Journal of Sound and Vibration* **201**, 153–167. The transient amplitude of the viscoelastic traveling string: an integral constitutive law.
14. A. H. Nayfeh and S. A. Nayfeh 1994 *Journal of Vibration and Acoustics* **116**, 129–136. On nonlinear modes of continuous systems.